

Competishun

CBSE Class 12 — Mathematics (041)

Sample Paper: Questions and Solutions — 2026-27 | Maximum Marks: 80 | Time Allowed: 3 Hours

GENERAL INSTRUCTIONS

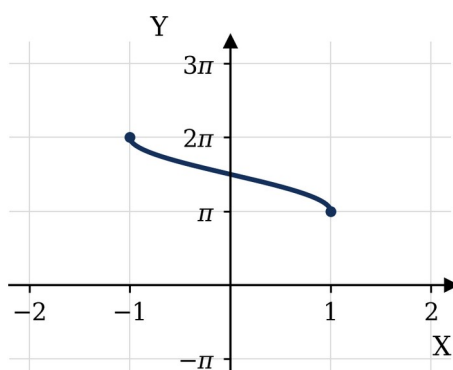
1. This question paper contains 38 questions. All questions are compulsory.
2. This question paper is divided into five Sections — A, B, C, D and E.
3. In Section A, Question no. 1 to 18 are multiple choice questions (MCQs) and Questions no. 19 and 20 are Assertion-Reason based questions of 1 mark each.
4. In Section B, Question no. 21 to 25 are Very Short Answer (VSA)-type questions, carrying 2 marks each.
5. In Section C, Question no. 26 to 31 are Short Answer (SA)-type questions, carrying 3 marks each.
6. In Section D, Question no. 32 to 35 are Long Answer (LA)-type questions, carrying 5 marks each.
7. In Section E, Question no. 36 to 38 are case study-based questions carrying 4 marks each.
8. There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and one sub-part each in 2 questions of Section E.
9. Draw neat and clean figures wherever required. Take $\pi = \frac{22}{7}$ wherever required, if not stated.
10. Use of calculators is not allowed.

SECTION A (20 × 1 = 20)

18 multiple choice questions and two assertion-reason type questions of 1 mark each. In every question of this section the correct option carries 1 mark; there are no part marks.

Question 1 — 1 mark

Identify the function whose graph is given below:



- (A) $\cos^{-1}\left(\frac{x}{2}\right)$ (B) $\cos^{-1}(2x)$ (C) $\pi - \cos^{-1}x$ (D) $\pi + \cos^{-1}x$

For Visually Impaired

Find the range of $\frac{\pi}{2} + \sec^{-1}\left(\frac{x}{2}\right)$

- (A) $[0, \pi]$ (B) $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ (C) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ (D) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$

ANSWER AND SOLUTION

(D) $\pi + \cos^{-1} x$

The graph is defined only on $[-1, 1]$, decreases throughout, and runs from 2π at $x = -1$ to π at $x = 1$.

Since $\cos^{-1} x$ decreases from π to 0 on $[-1, 1]$, the curve shown is $y = \pi + \cos^{-1} x$, whose range is $[\pi, 2\pi]$.

For Visually Impaired: (D) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$ — as $\sec^{-1} t \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$, adding $\frac{\pi}{2}$ shifts the set to $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$.

Question 2 — 1 mark

Domain of the function $\sec^{-1}(2x)$ is

(A) $(-\infty, \infty)$ (B) $\left(-\frac{1}{2}, \frac{1}{2}\right)$ (C) $\left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$ (D) $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$

ANSWER AND SOLUTION

(D) $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$

$\sec^{-1} t$ is defined for $|t| \geq 1$. Here $t = 2x$, so $|2x| \geq 1 \Rightarrow |x| \geq \frac{1}{2}$, which gives $x \leq -\frac{1}{2}$ or $x \geq \frac{1}{2}$. The end points are included.

Question 3 — 1 mark

Value of $\cot^{-1}\left(2 \cos\left(\frac{1}{2} \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right)$

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

ANSWER AND SOLUTION

(A) $\frac{\pi}{6}$

$$\cot^{-1}\left(2 \cos\left(\frac{1}{2} \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right) = \cot^{-1}\left(2 \cos\left(\frac{1}{2} \times \frac{\pi}{3}\right)\right) = \cot^{-1}\left(2 \cos \frac{\pi}{6}\right) = \cot^{-1}\left(2 \times \frac{\sqrt{3}}{2}\right) = \cot^{-1} \sqrt{3} = \frac{\pi}{6}$$

Question 4 — 1 mark

If $A = \begin{bmatrix} 2 & 1 & -2 \\ 3a & 0 & 4 \\ b & 2c & 1 \end{bmatrix}$ is a symmetric matrix, then value of $\sqrt{6a+2b+3c}$ is:

(A) -2 (B) 2 (C) ± 2 (D) 4

ANSWER AND SOLUTION

(B) 2

For a symmetric matrix $a_{ij} = a_{ji}$, so $3a = 1$, $b = -2$ and $2c = 4$, giving $6a = 2$, $b = -2$, $c = 2$.

Hence $\sqrt{6a+2b+3c} = \sqrt{2-4+6} = \sqrt{4} = 2$ (the radical sign denotes the non-negative root, so ± 2 is not accepted).

Question 5 — 1 mark

If P and Q are matrices of same order such that $P = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $|Q| = 3$, then find $|adj(PQ)|$

- (A) -3 (B) 1 (C) 6 (D) 36

ANSWER AND SOLUTION

- (C) 6

$|adj(PQ)| = |adj Q \cdot adj P| = |adj Q| |adj P| = |Q| |P|$ for matrices of order 2, since $|adj M| = |M|^{n-1} = |M|$ when $n = 2$.

Here $|P| = 2$ and $|Q| = 3$, so $|adj(PQ)| = 3 \times 2 = 6$.

Question 6 — 1 mark

If $\begin{bmatrix} 1 & x \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = O$, then value of x is

- (A) -2 (B) -1 (C) 1 (D) 2

ANSWER AND SOLUTION

- (D) 2

$$\begin{bmatrix} 1 & x \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2x \\ 0 & 4 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 2x \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 - 2x \\ 0 \end{bmatrix} = O.$$

Therefore $4 - 2x = 0 \Rightarrow x = 2$.

Question 7 — 1 mark

The derivative of $\cos^{-1}(2x^2 - 1)$ with respect to $\cos^{-1}(x)$ is

- (A) 2 (B) ± 1 (C) $\frac{1}{2}$ (D) -3

ANSWER AND SOLUTION

- (A) 2

Let $y = \cos^{-1}(2x^2 - 1)$ and $u = \cos^{-1}x$, so that $x = \cos u$.

Then $y = \cos^{-1}(2\cos^2 u - 1) = \cos^{-1}(\cos 2u) = 2u$, and hence $\frac{dy}{du} = 2$.

Question 8 — 1 mark

If a, b, c are arbitrary constants such that the function defined by $f(x) = \begin{cases} ax^2 + b, & b \neq 0, x \leq 1 \\ bx^2 + ax + c, & x > 1 \end{cases}$ is differentiable at $x = 1$, then which of the following could be a triplet for (a, b, c) ?

- (A) $(3, 2, 0)$ (B) $(0, 0, 3)$ (C) $\left(\frac{3}{2}, 2, 0\right)$ (D) $\left(3, \frac{3}{2}, 0\right)$

ANSWER AND SOLUTION

- (D) $\left(3, \frac{3}{2}, 0\right)$

Differentiability at $x = 1$ implies continuity at $x = 1$: $a + b = b + a + c \Rightarrow c = 0$.

Equality of the one-sided derivatives gives $2a = 2b + a \Rightarrow a = 2b$.

So (a, b, c) must be of the form $(2b, b, 0)$ with $b \neq 0$, which is satisfied only by option (D).

Question 9 — 1 mark

If $y = a \left\{ x + 2a \sin \left(\frac{x}{a} \right) \right\}$, $a > 0$, $x \in [0, \pi a]$; is a decreasing function, then the interval in which x lies is

- (A) $\left[\frac{5\pi}{6}, \pi \right]$ (B) $\left[\frac{2\pi}{3}, \pi \right]$ (C) $\left[\frac{5\pi a}{6}, \pi a \right]$ (D) $\left[\frac{2\pi a}{3}, \pi a \right]$

ANSWER AND SOLUTION

- (D) $\left[\frac{2\pi a}{3}, \pi a \right]$

y is decreasing $\Rightarrow \frac{dy}{dx} < 0 \Rightarrow a \left\{ 1 + 2 \cos \left(\frac{x}{a} \right) \right\} < 0 \Rightarrow \cos \left(\frac{x}{a} \right) < -\frac{1}{2}$.

Since $x \in [0, \pi a]$, we have $\frac{x}{a} \in [0, \pi]$, so $\frac{x}{a} \in \left[\frac{2\pi}{3}, \pi \right]$, that is $x \in \left[\frac{2\pi a}{3}, \pi a \right]$.

Question 10 — 1 mark

The values of x for which the derivative of the function $y = x^4$ is greater than the value of the function are

- (A) $0 < x < 4$ (B) $0 < x \leq 4$ (C) $x < 0$ or $x > 4$ (D) $x < 0$ or $x \geq 4$

ANSWER AND SOLUTION

- (A) $0 < x < 4$

$\frac{dy}{dx} > y \Rightarrow 4x^3 > x^4 \Rightarrow x^3(4 - x) > 0 \Rightarrow x(x - 4) < 0 \Rightarrow 0 < x < 4$.

Question 11 — 1 mark

The degree and order of the differential equation $y - x \left(\frac{dy}{dx} \right)^3 = a \left(y + \frac{d^2y}{dx^2} \right)^2$ are respectively

- (A) 3, 1 (B) 2, 2 (C) 3, 2 (D) 4, 2

ANSWER AND SOLUTION

- (B) 2, 2

The highest order derivative present is $\frac{d^2 y}{dx^2}$, so the order is 2. After expansion the equation is a polynomial in the derivatives and the term containing $\frac{d^2 y}{dx^2}$ carries the maximum power 2, so the degree is 2.

Question 12 — 1 mark

If $\int \frac{1}{1+e^{2x}} dx = x - P \log(1+e^{2x}) + C$, then P is equal to

- (A) $-\frac{1}{2}$ (B) $-\frac{1}{3}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$

ANSWER AND SOLUTION

- (D) $\frac{1}{2}$

$$\int \frac{1}{1+e^{2x}} dx = \int \left(\frac{1+e^{2x}}{1+e^{2x}} - \frac{e^{2x}}{1+e^{2x}} \right) dx = x - \frac{1}{2} \log|1+e^{2x}| + C.$$

Comparing with the given form, $P = \frac{1}{2}$.

Question 13 — 1 mark

If vector $2\hat{i} - u\hat{j} + 3\hat{k}$ has magnitude 4, then value of u is

- (A) 1 (B) $\sqrt{2}$ (C) $\pm\sqrt{3}$ (D) ± 2

ANSWER AND SOLUTION

- (C) $\pm\sqrt{3}$

$$|2\hat{i} - u\hat{j} + 3\hat{k}| = 4 \Rightarrow \sqrt{4+u^2+9} = 4 \Rightarrow u^2 = 16 - 13 = 3 \Rightarrow u = \pm\sqrt{3}.$$

Question 14 — 1 mark

Angle made by the vector $\hat{i} + \hat{j} + \sqrt{2}\hat{k}$ with positive direction of z -axis is

- (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

ANSWER AND SOLUTION

- (B) $\frac{\pi}{4}$

The magnitude of the vector is $\sqrt{1+1+2} = 2$, so its direction cosines are $\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}$.

If the required angle is γ , then $\cos \gamma = \frac{1}{\sqrt{2}} \Rightarrow \gamma = \frac{\pi}{4}$.

Question 15 — 1 mark

A vector perpendicular to the vectors $\hat{i} + \hat{j}$ and $\hat{i} - \hat{j}$ is

(A) $-2\hat{j}$ (B) $3\hat{k}$ (C) $2\hat{i}$ (D) $2\hat{i}+\hat{k}$

ANSWER AND SOLUTION

(B) $3\hat{k}$

Both $\hat{i}+\hat{j}$ and $\hat{i}-\hat{j}$ lie in the XY-plane, so a vector perpendicular to both must be along the z-axis. Indeed $(\hat{i}+\hat{j}) \times (\hat{i}-\hat{j}) = -2\hat{k}$, which is parallel to $3\hat{k}$.

Question 16 — 1 mark

The probability that it rains on a particular day is $\frac{1}{5}$. Assuming weather conditions of any two days are independent, find the probability that it rains on three consecutive days.

(A) $\frac{4}{5}$ (B) $\frac{1}{5}$ (C) $\frac{1}{25}$ (D) $\frac{1}{125}$

ANSWER AND SOLUTION

(D) $\frac{1}{125}$

By independence, the required probability $= \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} = \frac{1}{125}$.

Question 17 — 1 mark

A die is thrown twice. What is the probability that at least one number is 4, given the sum of the numbers appearing is a multiple of 5?

(A) $\frac{3}{5}$ (B) $\frac{4}{5}$ (C) $\frac{3}{7}$ (D) $\frac{4}{7}$

ANSWER AND SOLUTION

(D) $\frac{4}{7}$

Let A : the sum of the numbers appeared is a multiple of 5, and B : at least one number is 4.

$A = \{(1, 4), (2, 3), (3, 2), (4, 1), (4, 6), (5, 5), (6, 4)\}$, so $n(A) = 7$.

$A \cap B = \{(1, 4), (4, 1), (4, 6), (6, 4)\}$, so $n(A \cap B) = 4$.

Required probability $= P(B|A) = \frac{4}{7}$.

Question 18 — 1 mark

A and B are two independent events such that $P(A) = \frac{1}{6}$, $P(B) = k$. If the probability of occurrence of exactly one of them is $\frac{1}{3}$, then find the value of k .

(A) $\frac{1}{4}$ (B) $\frac{2}{3}$ (C) $\frac{3}{4}$ (D) $\frac{5}{6}$

ANSWER AND SOLUTION

(A) $\frac{1}{4}$

$$P(A' \cap B) + P(A \cap B') = \frac{1}{3} \Rightarrow P(A')P(B) + P(A)P(B') = \frac{1}{3}$$

$$\Rightarrow \left(1 - \frac{1}{6}\right)k + \frac{1}{6}(1 - k) = \frac{1}{3} \Rightarrow k\left(\frac{5}{6} - \frac{1}{6}\right) = \frac{1}{3} - \frac{1}{6} \Rightarrow \frac{4k}{6} = \frac{1}{6} \Rightarrow k = \frac{1}{4}.$$

ASSERTION-REASON BASED QUESTIONS

Questions 19 and 20 are Assertion and Reason based questions carrying 1 mark each. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the options given below:

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).
- (B) Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A).
- (C) Assertion (A) is true but Reason (R) is false.
- (D) Assertion (A) is false but Reason (R) is true.

Question 19 — 1 mark

Assertion: The value of the integral $\int_{-1}^1 |x - 2| dx$ is 4.

Reason: If $f(-x) = f(x)$ then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.

ANSWER AND SOLUTION

- (B) Both Assertion and Reason are true, but Reason is not the correct explanation of the Assertion.

On $[-1, 1]$ we have $x - 2 < 0$, so $|x - 2| = 2 - x$ and $\int_{-1}^1 |x - 2| dx = \int_{-1}^1 (2 - x) dx = \left[2x - \frac{x^2}{2}\right]_{-1}^1 = 4$. The

Assertion is true.

The Reason is a correct standard property, but $|x - 2|$ is not an even function, so the property is not used in evaluating this integral. Hence R does not explain A.

Question 20 — 1 mark

Assertion: The general solution of a differential equation represents a family of curves.

Reason: The derivative of an arbitrary constant is zero.

ANSWER AND SOLUTION

- (A) Both Assertion and Reason are true, and Reason is the correct explanation of the Assertion.

The general solution of a differential equation of order n contains n arbitrary constants; giving these constants different values produces different members of a family of curves. The constants survive the solution process precisely because the derivative of a constant is zero, so differentiating any member of the family returns the same differential equation. Hence R correctly explains A.

SECTION B (5 × 2 = 10)

5 very short answer (VSA) type questions of 2 marks each.

Question 21 — 2 marks

Q21 (A). Evaluate $\int \frac{1}{x\sqrt{x^6-1}} dx$

ANSWER AND SOLUTION — 21 (A)

$$I = \int \frac{1}{x\sqrt{x^6-1}} dx = \frac{1}{6} \int \frac{6x^5}{x^6\sqrt{x^6-1}} dx$$

Put $x^6 - 1 = t^2 \Rightarrow 6x^5 dx = 2t dt$ and $x^6 = t^2 + 1$.

$$I = \frac{1}{6} \int \frac{2t dt}{(t^2+1)t} = \frac{1}{3} \int \frac{dt}{t^2+1} = \frac{1}{3} \tan^{-1} t + C$$

$$I = \frac{1}{3} \tan^{-1}(\sqrt{x^6-1}) + C$$

OR

Q21 (B). Evaluate $\int \frac{\cos x - \sin x}{\sqrt{\sin 2x}} dx$

ANSWER AND SOLUTION — 21 (B)

$$I = \int \frac{\cos x - \sin x}{\sqrt{\sin 2x}} dx = \int \frac{\cos x - \sin x}{\sqrt{1 + \sin 2x - 1}} dx = \int \frac{\cos x - \sin x}{\sqrt{(\sin x + \cos x)^2 - 1}} dx$$

Put $\sin x + \cos x = t \Rightarrow (\cos x - \sin x) dx = dt$.

$$I = \int \frac{dt}{\sqrt{t^2-1}} = \log|t + \sqrt{t^2-1}| + C$$

$$I = \log|\sin x + \cos x + \sqrt{\sin 2x}| + C$$

MARKING SCHEME

21 (A): correct substitution — ½; reduction to $\frac{1}{3} \int \frac{dt}{t^2+1}$ and integration — 1; answer in terms of x with constant — ½.

21 (B): writing $\sqrt{\sin 2x} = \sqrt{(\sin x + \cos x)^2 - 1}$ — ½; substitution — ½; integration — ½; answer in terms of x with constant — ½.

Question 22 — 2 marks

If $y = \sin(x+y)$ and $\frac{dy}{dx} = 0$, then prove that $y = \pm 1$.

ANSWER AND SOLUTION

$y = \sin(x+y) \Rightarrow \sin^{-1} y = x+y$. Differentiating with respect to y ,

$$\frac{1}{\sqrt{1-y^2}} = \frac{dx}{dy} + 1 \Rightarrow \frac{dx}{dy} = \frac{1 - \sqrt{1-y^2}}{\sqrt{1-y^2}} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1 - \sqrt{1-y^2}}$$

$$\text{So } \frac{dy}{dx} = 0 \Rightarrow \sqrt{1-y^2} = 0 \Rightarrow y = \pm 1$$

Alternative way. $y = \sin(x+y) \Rightarrow \frac{dy}{dx} = \cos(x+y) \left(1 + \frac{dy}{dx}\right) \Rightarrow \frac{dy}{dx} (1 - \cos(x+y)) = \cos(x+y).$

Putting $\frac{dy}{dx} = 0$ gives $\cos(x+y) = 0 \Rightarrow \sin(x+y) = \pm 1 \Rightarrow y = \pm 1.$

MARKING SCHEME

Correct differentiation / correct relation for $\frac{dy}{dx} = 1$; using $\frac{dy}{dx} = 0$ to conclude $y = \pm 1$ — 1. Either method is accepted in full.

Question 23 — 2 marks

Q23 (A). A line passing through the points $(2, 6, 5)$ and $(3, p, 7)$ is parallel to the line joining the points $(-4, 3, 4)$ and $(-1, 0, q)$. Find the value of $\frac{q}{p}$.

ANSWER AND SOLUTION — 23 (A)

Given points are $A(2, 6, 5)$, $B(3, p, 7)$, $C(-4, 3, 4)$ and $D(-1, 0, q)$.

Direction ratios of $\overrightarrow{AB}$ are $\langle 1, p-6, 2 \rangle$ and direction ratios of $\overrightarrow{CD}$ are $\langle 3, -3, q-4 \rangle$.

As the lines are parallel, $\frac{1}{3} = \frac{p-6}{-3} = \frac{2}{q-4}$

$\Rightarrow p-6 = -1$ and $q-4 = 3 \times 2$, so $p=5$ and $q=10$.

Thus $\frac{q}{p} = \frac{10}{5} = 2$

OR

Q23 (B). A rectangle $ABCD$ is such that $\overrightarrow{AC} = \vec{u}$ and $\overrightarrow{BD} = \vec{v}$, then find the vectors representing four sides of the rectangle (in a cyclic order starting from $\overrightarrow{AB}$).

ANSWER AND SOLUTION — 23 (B)

Let the vectors representing the four sides be $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$, $\overrightarrow{CD} = \vec{c}$ and $\overrightarrow{DA} = \vec{d}$.

Then $\overrightarrow{AC} = \vec{u} = \vec{a} + \vec{b}$... (i) and $\overrightarrow{BD} = \vec{v} = \vec{b} + \vec{c} = \vec{b} - \vec{a}$... (ii), since $\vec{c} = -\vec{a}$.

Solving (i) and (ii): $\vec{a} = \frac{1}{2}(\vec{u} - \vec{v})$ and $\vec{b} = \frac{1}{2}(\vec{u} + \vec{v})$

Also $\vec{c} = -\vec{a} = \frac{1}{2}(\vec{v} - \vec{u})$ and $\vec{d} = -\vec{b} = -\frac{1}{2}(\vec{u} + \vec{v})$

MARKING SCHEME

23 (A): writing both sets of direction ratios and the condition of parallelism — 1; finding $p = 5$; finding q and $\frac{q}{p} = 2$.

23 (B): forming equations (i) and (ii) — $\frac{1}{2}$; solving for $\vec{a}$ and $\vec{b}$ — 1; writing $\vec{c}$ and $\vec{d}$ — $\frac{1}{2}$.



Question 24 — 2 marks

For two non-zero vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}| = \sqrt{6}$, $\vec{a} \cdot \vec{b} = 1$ and $\vec{b} \times \vec{a} = 3\hat{i} + 4\hat{j} - 2\hat{k}$. Find $|\vec{b}|$.

ANSWER AND SOLUTION

$$|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{a}| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{29}$$

Using the identity $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2$,

$$29 + 1 = 6|\vec{b}|^2 \Rightarrow |\vec{b}|^2 = \frac{30}{6} = 5$$

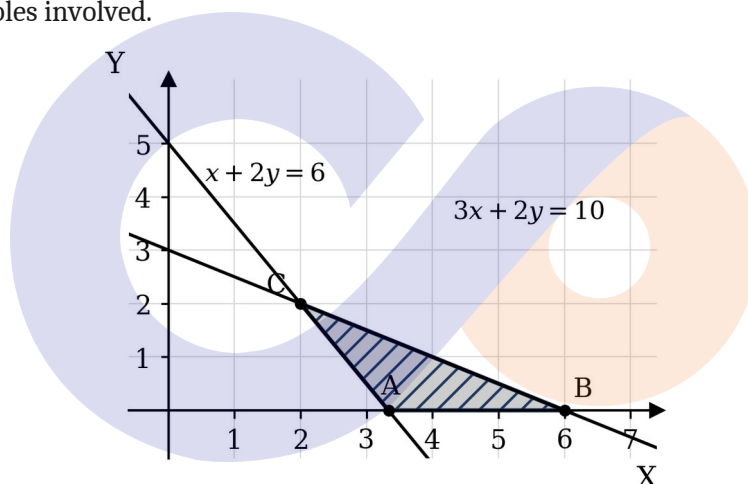
$$|\vec{b}| = \sqrt{5}$$

MARKING SCHEME

Finding $|\vec{b} \times \vec{a}| = \sqrt{29}$ — $\frac{1}{2}$; writing the correct identity — $\frac{1}{2}$; substitution and final answer — 1.

Question 25 — 2 marks

The feasible region of a linear programming problem is shown in the given figure. Write all the constraints on the decision variables involved.



For Visually Impaired

For a linear programming problem, the maximum value of an objective function $Z = 3x + 4y$ exists at the points (p, q) and $(2p, \frac{q}{2})$. Find the relation between p and q .

ANSWER AND SOLUTION

The shaded region lies on or above the x -axis, on or to the right of the line $3x + 2y = 10$, and on or below the line $x + 2y = 6$. Hence the constraints are

$$3x + 2y \geq 10, x + 2y \leq 6, y \geq 0$$

For Visually Impaired. As the maximum value exists at both (p, q) and $(2p, \frac{q}{2})$, the objective function takes the same value at these points:

$$3p + 4q = 3(2p) + 4\left(\frac{q}{2}\right) \Rightarrow 3p + 4q = 6p + 2q \Rightarrow 6p - 3p = 4q - 2q$$

$$3p = 2q$$

MARKING SCHEME

Writing the two line constraints with correct inequality signs — $1\frac{1}{2}$; writing $y \geq 0$ — $\frac{1}{2}$. **For Visually Impaired:** equating the two values of Z — 1; simplifying to $3p = 2q - 1$.

SECTION C ($6 \times 3 = 18$)

6 short answer (SA) type questions of 3 marks each.

Question 26 — 3 marks

For two matrices $P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$, find PQ and QP , whichever is defined. Also, find the inverse of the product matrix, if it exists.

ANSWER AND SOLUTION

Both products are defined. Let $A = PQ$ and $B = QP$. Then

$$A = PQ = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

$$B = QP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 4 \\ 2 & 0 & 0 \end{bmatrix}$$

Here $|A| = 4 \neq 0$, so A is non-singular and its inverse exists:

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{4} \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Here $|B| = 0$ (its first and third rows are identical), so B is singular and $(QP)^{-1}$ does not exist.

MARKING SCHEME

Correct PQ — 1; correct QP — 1; $|A| = 4$ and A^{-1} — $\frac{1}{2}$; $|B| = 0$ with the conclusion that B^{-1} does not exist — $\frac{1}{2}$.

Question 27 — 3 marks

The circular area of vision on the ground by a surveillance drone fitted with camera increases at the rate of 44 m^2 per metre as it gains vertical height. The radius of the circle on the ground increases at the rate of 6 m/s. Find the rate at which the height of the drone increases with time, at the instant when the radius of the circular area on the ground is 14 m. (Use $\pi = \frac{22}{7}$)

ANSWER AND SOLUTION

Let A be the area of the circular region covered by the drone, r its radius and h the height of the drone at time t .

Given $\frac{dA}{dh} = 44 \text{ m}^2$ per metre and $\frac{dr}{dt} = 6 \text{ m/s}$; to find $\frac{dh}{dt}$ when $r = 14 \text{ m}$.

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \Rightarrow \frac{dA}{dh} \cdot \frac{dh}{dt} = 2\pi r \frac{dr}{dt}$$

$$44 \frac{dh}{dt} = 2 \left(\frac{22}{7} \right) (14)(6) = 528 \Rightarrow \frac{dh}{dt} \Big|_{r=14} = 12$$

The height of the drone increases at the rate of 12 m/s at that moment.

MARKING SCHEME

Writing the given rates and what is to be found — $\frac{1}{2}$; relating $\frac{dA}{dt}$, $\frac{dA}{dh}$ and $\frac{dh}{dt}$ with correct differentiation — $1\frac{1}{2}$; substitution and final answer with unit — 1.

Question 28 — 3 marks

Q28 (A). If $x = \sin^3 t$, $y = \cos^3 t$, evaluate $\frac{d^2 y}{dx^2}$ at $\left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}} \right)$.

ANSWER AND SOLUTION — 28 (A)

$$\frac{dx}{dt} = 3\sin^2 t \cos t \text{ and } \frac{dy}{dt} = -3\cos^2 t \sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\cot t$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx} = \operatorname{cosec}^2 t \left(\frac{1}{3\sin^2 t \cos t} \right) = \frac{1}{3} \sec t \cdot \operatorname{cosec}^4 t$$

$$\text{At the point } \left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}} \right), \sin^3 t = \frac{1}{2\sqrt{2}} \Rightarrow \sin t = \frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4}$$

$$\left. \frac{d^2 y}{dx^2} \right|_{t=\pi/4} = \frac{1}{3} (\sqrt{2})(\sqrt{2})^4 = \frac{4\sqrt{2}}{3}$$

OR

Q28 (B). If $y = x^{\log x} + (\log x)^x$; $x > 1$, then evaluate $\frac{dy}{dx}$.

ANSWER AND SOLUTION — 28 (B)

Let $y = x^{\log x} + (\log x)^x = u + v$, where $u = x^{\log x}$ and $v = (\log x)^x$.

$$\log u = (\log x)^2 \text{ and } \log v = x \log(\log x)$$

$$\frac{1}{u} \frac{du}{dx} = 2(\log x) \cdot \frac{1}{x} \text{ and } \frac{1}{v} \frac{dv}{dx} = \log(\log x) + \frac{1}{\log x}$$

$$\frac{du}{dx} = x^{\log x} \left(\frac{2 \log x}{x} \right) \text{ and } \frac{dv}{dx} = (\log x)^x \left(\log(\log x) + \frac{1}{\log x} \right)$$

$$\frac{dy}{dx} = \frac{2}{x} x^{\log x} \log x + (\log x)^x \left(\log(\log x) + \frac{1}{\log x} \right)$$

MARKING SCHEME

28 (A): $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx} = 1$; correct $\frac{d^2y}{dx^2} = 1$; finding $t = \frac{\pi}{4}$ and the value $= 1$.

28 (B): splitting as $u + v$ and taking logarithms $= 1$; $\frac{du}{dx} = 1$; $\frac{dv}{dx}$ and the final sum $= 1$. (Writing $u = e^{(\log x)^2}$, $v = e^{x \log(\log x)}$ is an equally acceptable method.)

Question 29 — 3 marks

Q29 (A). Find the particular solution of the differential equation $x dy + (y - x^2 e^x) dx = 0$, given that $y(1) = 0$.

ANSWER AND SOLUTION — 29 (A)

The equation is linear: $\frac{dy}{dx} + \frac{1}{x}y = x e^x$

$$\text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Solution is $y \cdot x = \int x^2 e^x dx$. Integrating by parts twice,

$$yx = x^2 e^x - \int 2x e^x dx = x^2 e^x - 2(x e^x - e^x) + C$$

$$y(1) = 0 \Rightarrow 0 = e - 2(e - e) + C \Rightarrow C = -e$$

$$yx = e^x (x^2 - 2x + 2) - e$$

OR

Q29 (B). Find the particular solution of the differential equation $(x^2 + y^2) dy = x y dx$, given that $y(0) = 1$.

ANSWER AND SOLUTION — 29 (B)

$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$ is a homogeneous differential equation.

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$. The equation becomes

$$v + x \frac{dv}{dx} = \frac{v}{1+v^2} \Rightarrow x \frac{dv}{dx} = \frac{v - v - v^3}{1+v^2} = \frac{-v^3}{1+v^2}$$

$$\int \frac{1+v^2}{v^3} dv = - \int \frac{dx}{x} \Rightarrow \int \left(\frac{1}{v^3} + \frac{1}{v} \right) dv = - \log|x| + \log C$$

$$\frac{-1}{2v^2} + \log|v| = - \log|x| + \log C \Rightarrow \log \left| \frac{y}{C} \right| = \frac{x^2}{2y^2}, \text{ that is } y = C e^{x^2/(2y^2)}$$

Putting $x=0$, $y=1$ gives $C=1$.

$$y = e^{x^2/(2y^2)}$$

MARKING SCHEME

29 (A): reducing to linear form $= 1/2$; integrating factor $= 1/2$; integration by parts $= 1$; evaluating $C = 1/2$; particular solution $= 1/2$.

29 (B): substitution $y = vx$ and separation of variables $= 1$; integration $= 1$; evaluating C and writing the particular solution $= 1$.



Question 30 — 3 marks**Q30 (A).** Find the foot of the perpendicular drawn from the point $(2, 1, 4)$ on the line $x - 2 = \frac{2y+3}{4} = 2 - z$. Also find the equation of the line passing through this point and the point $(2, 1, 4)$.**ANSWER AND SOLUTION — 30 (A)**

A point on the line $x - 2 = \frac{2y+3}{4} = 2 - z = k$ is $P\left(k+2, \frac{4k-3}{2}, 2-k\right)$, and the direction ratios of the given line are $\langle 1, 2, -1 \rangle$.

Direction ratios of the line joining $A(2, 1, 4)$ and P are $\left\langle k, \frac{4k-5}{2}, -k-2 \right\rangle$.

As AP is perpendicular to the given line,

$$1 \times k + 2 \times \frac{4k-5}{2} + (-1) \times (-k-2) = 0 \Rightarrow k + 4k - 5 + k + 2 = 0 \Rightarrow 6k - 3 = 0 \Rightarrow k = \frac{1}{2}$$

So the foot of the perpendicular is $P\left(\frac{5}{2}, -\frac{1}{2}, \frac{3}{2}\right)$

Direction ratios of AP are $\left\langle \frac{1}{2}, -\frac{3}{2}, -\frac{5}{2} \right\rangle$, that is $\langle 1, -3, -5 \rangle$, and the required line is

$$\frac{x-2}{1} = \frac{y-1}{-3} = \frac{z-4}{-5}, \text{ or equivalently } x-2 = \frac{1-y}{3} = \frac{4-z}{5}$$

OR

Q30 (B). Find the angle between the lines $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-2}{1}$ and $\frac{x-5}{2} = y - \frac{1}{2} = \frac{3-z}{2}$. Also find the cartesian equation of a line passing through the point $(1, 1, 1)$ and perpendicular to both lines.

ANSWER AND SOLUTION — 30 (B)

The second line can be written as $\frac{x-5}{2} = \frac{y-\frac{1}{2}}{1} = \frac{z-3}{-2}$, so the direction ratios of the two lines are

$\langle 2, 2, 1 \rangle$ and $\langle 2, 1, -2 \rangle$.

If θ is the angle between the lines, $\cos \theta = \frac{2 \times 2 + 2 \times 1 + 1 \times (-2)}{\sqrt{9} \times \sqrt{9}} = \frac{4}{9}$

$$\theta = \cos^{-1}\left(\frac{4}{9}\right)$$

The required line is along $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ 2 & 1 & -2 \end{vmatrix} = -5\hat{i} + 6\hat{j} - 2\hat{k}$

$$\frac{x-1}{-5} = \frac{y-1}{6} = \frac{z-1}{-2}, \text{ or equivalently } \frac{x-1}{5} = \frac{y-1}{-6} = \frac{z-1}{2}$$

MARKING SCHEME

30 (A): general point on the line and its direction ratios — $\frac{1}{2}$; direction ratios of AP — $\frac{1}{2}$; perpendicularity condition — $\frac{1}{2}$; value of k — $\frac{1}{2}$; foot of perpendicular — $\frac{1}{2}$; equation of the line — $\frac{1}{2}$.

30 (B): writing both sets of direction ratios in standard form — $\frac{1}{2}$; $\cos \theta = \frac{4}{9}$ and the angle — 1 ; cross product — 1 ; equation of the required line — $\frac{1}{2}$.

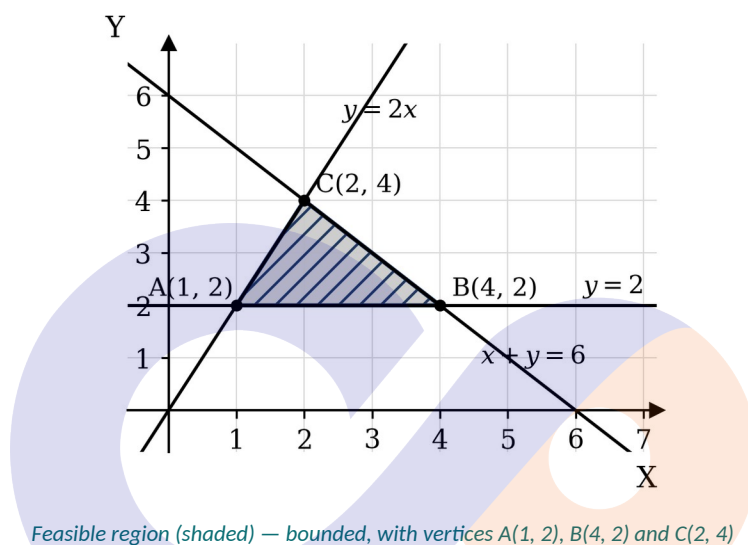
Question 31 — 3 marks

Solve the linear programming problem graphically: Maximize $Z = 40x + 30y$ subject to the constraints $x + y \leq 6$, $2x \geq y$, $y \geq 2$.

For Visually Impaired

For a linear programming problem, vertices of the feasible region are $(20, 0)$, $(40, 0)$, $(30, 10)$, $(0, 20)$ and $(0, 10)$. Minimize and maximize the objective function $Z = 50x - 20y$.

ANSWER AND SOLUTION



The feasible region is bounded with the three vertices $A(1, 2)$, $B(4, 2)$ and $C(2, 4)$.

$$Z_A = 40(1) + 30(2) = 100, Z_B = 40(4) + 30(2) = 220, Z_C = 40(2) + 30(4) = 200$$

Maximum value of Z is 220 at the point $B(4, 2)$.

For Visually Impaired. With $Z = 50x - 20y$ at the vertices $A(20, 0)$, $B(40, 0)$, $C(30, 10)$, $D(0, 20)$, $E(0, 10)$:

$$Z_A = 1000, Z_B = 2000, Z_C = 1500 - 200 = 1300, Z_D = -400, Z_E = -200$$

Minimum value of Z is -400 at $D(0, 20)$ and maximum value of Z is 2000 at $B(40, 0)$.

MARKING SCHEME

Correct graph with the shaded feasible region and its vertices — $\frac{1}{2}$; values of Z at the corner points — 1 ; maximum value with the point at which it occurs — $\frac{1}{2}$. **For Visually Impaired:** values of Z at the five vertices — 2 ; minimum and maximum with their points — 1 .

SECTION D ($4 \times 5 = 20$)

4 long answer (LA) type questions of 5 marks each.

Question 32 — 5 marks

Q32 (A). Let $A = N \times N$, where N is the set of natural numbers and a relation R is defined on A such that $(a, b) R (c, d)$ if and only if $a^2 + d^2 = b^2 + c^2$. Check whether R is an equivalence relation or not.

ANSWER AND SOLUTION — 32 (A)

Reflexive. Let $(p, q) \in A = N \times N$. As $p, q \in N$, $p^2 + q^2 = q^2 + p^2$, so $(p, q) R (p, q)$ for all $(p, q) \in N \times N$. Hence R is reflexive.

Symmetric. Let $(p, q), (r, s) \in A$ with $(p, q) R (r, s)$. Then $p^2 + s^2 = q^2 + r^2$
 $\Rightarrow q^2 + r^2 = p^2 + s^2 \Rightarrow r^2 + q^2 = s^2 + p^2 \Rightarrow (r, s) R (p, q)$. Hence R is symmetric.

Transitive. Let $(p, q), (r, s), (u, v) \in A$ with $(p, q) R (r, s)$ and $(r, s) R (u, v)$. Then
 $p^2 + s^2 = q^2 + r^2 \dots$ (i) and $r^2 + v^2 = s^2 + u^2 \dots$ (ii)

Adding (i) and (ii): $p^2 + s^2 + r^2 + v^2 = q^2 + r^2 + s^2 + u^2 \Rightarrow p^2 + v^2 = q^2 + u^2 \Rightarrow (p, q) R (u, v)$. Hence R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation.

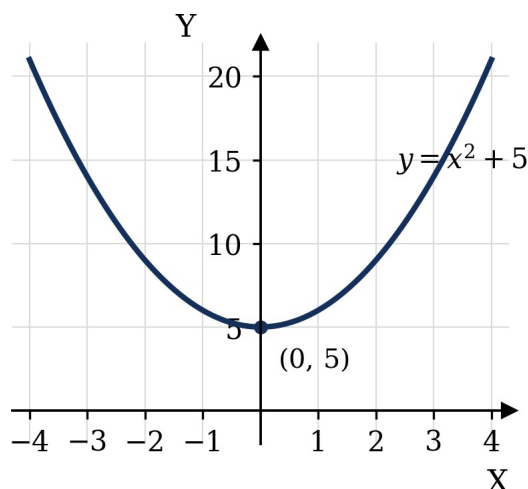
OR

Q32 (B). A relation $R: (-\infty, \infty) \rightarrow [0, \infty)$ is defined as $R = \{(x, y) : y = x^2 + 5\}$. Draw its graph and check whether it is a function or not. If the given relation is a function, then examine whether the function is injective, and whether it is surjective or not.

For Visually Impaired

A relation $R: (-\infty, \infty) \rightarrow [0, \infty)$ is defined as $R = \{(x, y) : y = x^2 + 7\}$. Check whether it is a function or not. If the given relation is a function, then examine whether the function is injective, surjective, or both, and conclude whether it is bijective or not.

ANSWER AND SOLUTION — 32 (B)



Graph of $y = x^2 + 5$ — a parabola with vertex $(0, 5)$, opening upwards

Yes, R is a function, as the graph shows that every element of the set $(-\infty, \infty)$ has a unique image in the set $[0, \infty)$.

Denote this function by $f: (-\infty, \infty) \rightarrow [0, \infty)$, $f(x) = x^2 + 5$.

Injective. $f(-1) = (-1)^2 + 5 = 6$ and $f(1) = 1^2 + 5 = 6$, so $f(-1) = f(1) = 6$ while $-1 \neq 1$. Thus f is many-one, so it is not injective.

Surjective. Range of $f = [5, \infty) \neq [0, \infty) = \text{co-domain}$. For instance, $y = 1$ has no pre-image, since $1 = x^2 + 5 \Rightarrow x^2 = -4$, which is impossible. Thus it is not surjective.

For Visually Impaired. For every element a of $(-\infty, \infty)$, $a^2 \in [0, \infty)$ and therefore $a^2 + 7 \in [7, \infty)$. Hence every element of $(-\infty, \infty)$ has a unique image in $[0, \infty)$, so $f: (-\infty, \infty) \rightarrow [0, \infty)$, $f(x) = x^2 + 7$ is a function.

$f(-1) = 8 = f(1)$, so f is many-one and not injective. Range of $f = [7, \infty) \neq [0, \infty)$, so f is not surjective. Hence f is neither injective nor surjective, and therefore not bijective.

MARKING SCHEME

32 (A): reflexive — $1\frac{1}{2}$; symmetric — $1\frac{1}{2}$; transitive — $1\frac{1}{2}$; conclusion — $\frac{1}{2}$.

32 (B): correct graph — 1; stating that R is a function — 1; injectivity examined — $1\frac{1}{2}$; surjectivity examined — $1\frac{1}{2}$. **For Visually Impaired:** showing it is a function — 1; injectivity — $1\frac{1}{2}$; surjectivity — $1\frac{1}{2}$; conclusion about bijectivity — 1.

Question 33 — 5 marks

Evaluate $\int_{-a}^a \sqrt{\frac{a-x}{a+x}} dx$

ANSWER AND SOLUTION

$$\begin{aligned} I &= \int_{-a}^a \sqrt{\frac{a-x}{a+x}} dx = \int_{-a}^a \sqrt{\frac{a-x}{a+x} \times \frac{a-x}{a-x}} dx = \int_{-a}^a \frac{a-x}{\sqrt{a^2-x^2}} dx \\ &= \int_{-a}^a \frac{a}{\sqrt{a^2-x^2}} dx - \int_{-a}^a \frac{x}{\sqrt{a^2-x^2}} dx = I_1 - I_2 \end{aligned}$$

Now $\frac{a}{\sqrt{a^2-x^2}}$ is an even function, so

$$I_1 = 2a \int_0^a \frac{dx}{\sqrt{a^2-x^2}} = 2a \left[\sin^{-1} \frac{x}{a} \right]_0^a = 2a \left(\frac{\pi}{2} \right) = \pi a$$

and $\frac{x}{\sqrt{a^2-x^2}}$ is an odd function, so $I_2 = 0$.

Hence $I = I_1 - I_2 = \pi a$

Alternate method. Let $x = a \cos 2\theta \Rightarrow dx = -2a \sin 2\theta d\theta$. When $x = -a$, $\theta = \frac{\pi}{2}$ and when $x = a$, $\theta = 0$.

$$I = -2a \int_{\pi/2}^0 \tan \theta \sin 2\theta d\theta = 2a \int_0^{\pi/2} \tan \theta \sin 2\theta d\theta \dots (i)$$

$$\text{Using } \int_0^a f(x) dx = \int_0^a f(a-x) dx,$$

$$I = 2a \int_0^{\pi/2} \cot \theta \sin 2\theta d\theta \dots (ii)$$

$$\text{Adding (i) and (ii): } 2I = 2a \int_0^{\pi/2} (\tan \theta + \cot \theta) \sin 2\theta d\theta = 4a \int_0^{\pi/2} (\sin^2 \theta + \cos^2 \theta) d\theta$$

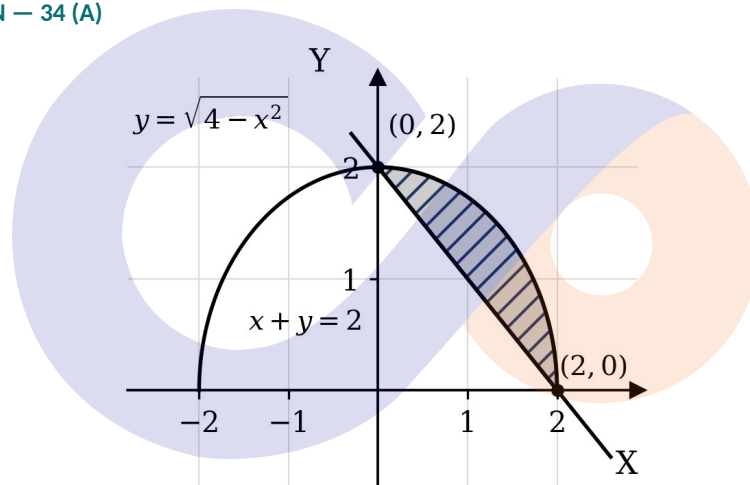
$$\Rightarrow I = 2a \int_0^{\pi/2} d\theta = 2a \left(\frac{\pi}{2} \right) = \pi a$$

MARKING SCHEME

Rationalising to $\int \frac{a-x}{\sqrt{a^2-x^2}} dx - \frac{1}{2}$; splitting into I_1 and $I_2 - 1$; evaluating $I_1 = \pi a$ using the even-function property — 2; stating $I_2 = 0$ using the odd-function property — 1; final answer — $\frac{1}{2}$. The substitution method shown above earns full marks.

Question 34 — 5 marks

Q34 (A). Using integration, find the area of the region $\{(x, y) : 0 \leq y \leq \sqrt{4-x^2}, x \geq 2-y\}$.

ANSWER AND SOLUTION — 34 (A)

$y = \sqrt{4-x^2}$ is a semi-circle of radius 2 units with centre $(0, 0)$, and $x = 2 - y$, that is $x + y = 2$, is a line with intercept 2 on each axis. The curves meet at $(0, 2)$ and $(2, 0)$.

$$\text{Area} = \int_0^2 (\sqrt{4-x^2} - (2-x)) dx = \left[\frac{x}{2} \sqrt{4-x^2} + 2 \sin^{-1} \frac{x}{2} - 2x + \frac{x^2}{2} \right]_0^2$$

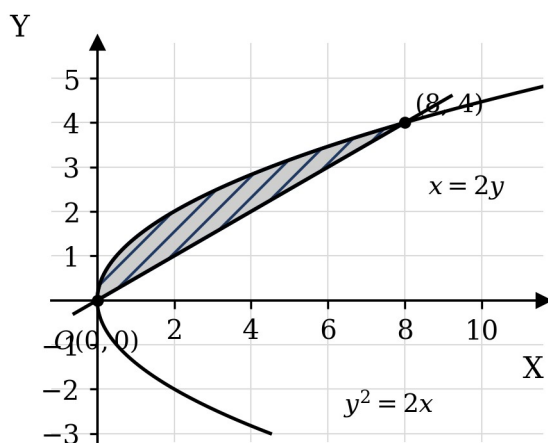
$$= \left(0 + 2 \cdot \frac{\pi}{2} - 4 + 2 \right) - 0 = \pi - 2$$

Area = $(\pi - 2)$ sq. units

OR

Q34 (B). Using integration, find the area of the region bounded by the curve $y^2 = 2x$ and the line $x - 2y = 0$.

ANSWER AND SOLUTION — 34 (B)



$y^2 = 2x$ represents a parabola whose points of intersection with the line $x = 2y$ are $(0, 0)$ and $(8, 4)$.

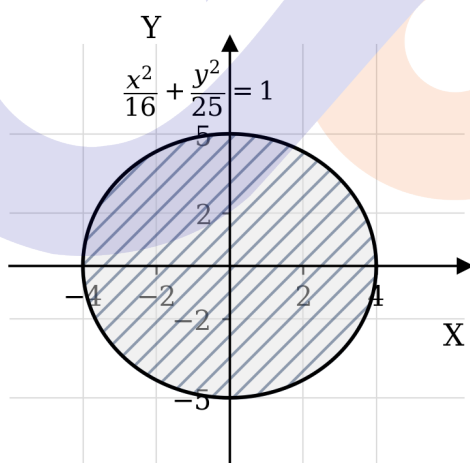
$$\text{Area} = \int_0^8 \left(\sqrt{2x} - \frac{x}{2} \right) dx = \left[\sqrt{2} \cdot \frac{2}{3} x^{3/2} - \frac{x^2}{4} \right]_0^8 = \frac{64}{3} - 16 = \frac{16}{3}$$

Area = $\frac{16}{3}$ sq. units

For Visually Impaired — Q34 (A)

Using integration, find the area of the region enclosed within $25x^2 + 16y^2 = 400$.

ANSWER AND SOLUTION



$25x^2 + 16y^2 = 400$ is an ellipse with standard equation $\frac{x^2}{16} + \frac{y^2}{25} = 1$, so $y = \frac{5}{4} \sqrt{4^2 - x^2}$.

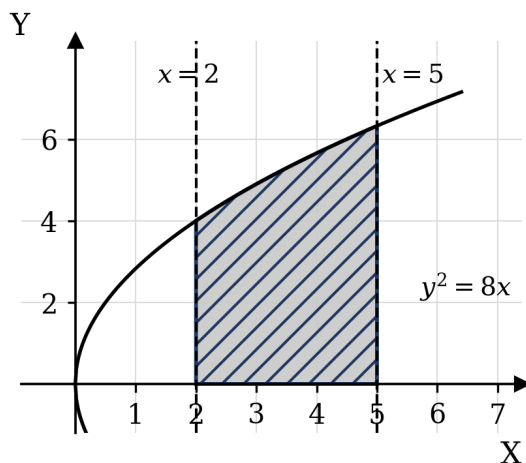
$$\text{Area} = 4 \int_0^4 \frac{5}{4} \sqrt{4^2 - x^2} dx = 5 \int_0^4 \sqrt{16 - x^2} dx = 5 \left[\frac{x}{2} \sqrt{16 - x^2} + 8 \sin^{-1} \frac{x}{4} \right]_0^4 = 5[4\pi]$$

Area = 20π sq. units

For Visually Impaired — Q34 (B)

Using integration, find the area of the region in the first quadrant bounded by the curve $y^2 = 8x$ and the lines $x = 2$, $x = 5$ and the x -axis.

ANSWER AND SOLUTION



$y^2 = 8x$ represents a parabola, and the required area lies above the x -axis, below the parabola, between the lines $x=2$ and $x=5$.

$$\text{Area} = \int_2^5 \sqrt{8x} \, dx = \left[2\sqrt{2} \cdot \frac{2}{3} x^{3/2} \right]_2^5 = \frac{4\sqrt{2}}{3} (5\sqrt{5} - 2\sqrt{2}) = \frac{4}{3} (5\sqrt{10} - 4)$$

$$\text{Area} = \frac{4}{3} (5\sqrt{10} - 4) \text{ sq. units} \approx 15.75 \text{ sq. units}$$

MARKING SCHEME

34 (A): correct figure — 1; identifying the curves and points of intersection — 1; correct integral with limits — 1; integration — 1; final answer with units — 1.

34 (B): correct figure — 1; points of intersection — 1; correct integral with limits — 1; integration — 1; final answer with units — 1. **For Visually Impaired (A):** standard form — $\frac{1}{2}$; integral set up — 1; use of symmetry — 1; integration — $1\frac{1}{2}$; answer — 1. **For Visually Impaired (B):** integral with limits — $\frac{1}{2}$; integration — 2; simplification — $1\frac{1}{2}$; answer — 1.

Question 35 — 5 marks

A fitness app syncs with four types of smartwatches: Alpha, Beta, Gamma and Delta. 45% of the users use Alpha, 30% use Beta, 15% use Gamma and 10% use Delta. Due to device errors, the probabilities of getting a false heart-rate alert from the Alpha, Beta, Gamma and Delta smartwatches are respectively 0.01, 0.02, 0.05 and 0.01.

Based on the given information, answer the following questions:

- Find the probability that a randomly selected user gets a false alert.
- If a false alert occurs, what is the probability that it came from a Beta smartwatch?

ANSWER AND SOLUTION

Let E_1, E_2, E_3, E_4 be the events that the user uses an Alpha, Beta, Gamma and Delta smartwatch respectively, and let A be the event that the user gets a false heart-rate alert. Clearly $\{E_1, E_2, E_3, E_4\}$ is a partition of the sample space.

$$P(E_1) = 0.45, P(E_2) = 0.3, P(E_3) = 0.15, P(E_4) = 0.1$$

$$P(A|E_1) = 0.01, P(A|E_2) = 0.02, P(A|E_3) = 0.05, P(A|E_4) = 0.01$$

(i) By the Theorem of Total Probability,

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)P(A|E_3) + P(E_4)P(A|E_4)$$

$$= 0.45 \times 0.01 + 0.3 \times 0.02 + 0.15 \times 0.05 + 0.1 \times 0.01 = 0.0045 + 0.006 + 0.0075 + 0.001$$

$$P(A) = 0.019$$

(ii) By Bayes' Theorem,

$$P(E_2|A) = \frac{P(E_2)P(A|E_2)}{P(A)} = \frac{0.3 \times 0.02}{0.019} = \frac{0.006}{0.019} = \frac{6}{19}$$

MARKING SCHEME

Defining the events and writing the prior probabilities — 1; writing the conditional probabilities — 1; part (i) using total probability — 1½; part (ii) using Bayes' Theorem — 1½.

SECTION E (3 × 4 = 12)

3 case-study / passage-based questions of 4 marks each with sub-parts. The first two have sub-parts (i), (ii), (iii) of 1, 1 and 2 marks; the third has two sub-parts of 2 marks each.

Question 36 — 1 + 1 + 2 = 4 marks

Two security sensors in a multi-storey mall emit laser beams for movement detection. Three of these beams travel along the following lines:

$$l_1: \vec{r} = (\lambda + 1)\hat{i} + (3\lambda + 2)\hat{j} + (3 - \lambda)\hat{k} \text{ and}$$

$$l_2: \vec{r} = 2(\mu + 1)\hat{i} + (3\mu - 1)\hat{j} + (\mu + 1)\hat{k}$$

$$l_3: \vec{r} = (\gamma + 3)\hat{i} + (3\gamma - 1)\hat{j} + (1 - \gamma)\hat{k}$$

Based on the given information, answer the following questions:

- (i) Show that the lines l_1 and l_2 are not parallel to each other.
- (ii) Show that the lines l_1 and l_3 are parallel to each other.
- (iii)
 - (a) Find the shortest distance between the lines l_1 and l_2 .

OR

- (iii)
 - (b) Find the shortest distance between the lines l_1 and l_3 .

ANSWER AND SOLUTION

(i) The given lines are

$$l_1: \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} + 3\hat{j} - \hat{k}) \equiv \vec{a} + \lambda\vec{b} \text{ and } l_2: \vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k}) \equiv \vec{c} + \mu\vec{d}$$

$$\text{Here } \vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = \hat{i} + 3\hat{j} - \hat{k}, \vec{c} = 2\hat{i} - \hat{j} + \hat{k}, \vec{d} = 2\hat{i} + 3\hat{j} + \hat{k}.$$

For the direction vectors $\vec{b}$ and $\vec{d}$, $\vec{b} \neq \lambda\vec{d}$ for any $\lambda \in \mathbb{R}$, so $\vec{b}$ is not parallel to $\vec{d}$. Hence the lines are not parallel.

$$(ii) l_3: \vec{r} = 3\hat{i} - \hat{j} + \hat{k} + \gamma(\hat{i} + 3\hat{j} - \hat{k}) \equiv \vec{e} + \gamma\vec{f}$$

Since $\vec{f} = \vec{b}$, the lines l_1 and l_3 are parallel.

$$(iii) (a) \vec{c} - \vec{a} = (2\hat{i} - \hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = \hat{i} - 3\hat{j} - 2\hat{k}$$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -1 \\ 2 & 3 & 1 \end{vmatrix} = 6\hat{i} - 3\hat{j} - 3\hat{k} = 3(2\hat{i} - \hat{j} - \hat{k})$$

$$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = (\hat{i} - 3\hat{j} - 2\hat{k}) \cdot 3(2\hat{i} - \hat{j} - \hat{k}) = 3(2 + 3 + 2) = 21 \neq 0, \text{ so the lines are skew.}$$

$$|\vec{b} \times \vec{d}| = 3\sqrt{2^2 + (-1)^2 + (-1)^2} = 3\sqrt{6}$$

$$D = \frac{|(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d})|}{|\vec{b} \times \vec{d}|} = \frac{|21|}{3\sqrt{6}} = \frac{7}{\sqrt{6}} = \frac{7\sqrt{6}}{6} \text{ units}$$

(iii) (b) The lines l_1 and l_3 are parallel, so the shortest distance is $\frac{|\vec{b} \times (\vec{e} - \vec{a})|}{|\vec{b}|}$, where $\vec{e} - \vec{a} = 2\hat{i} - 3\hat{j} - 2\hat{k}$.

$$\vec{b} \times (\vec{e} - \vec{a}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -1 \\ 2 & -3 & -2 \end{vmatrix} = -9\hat{i} - 9\hat{k}$$

$$\text{Shortest distance} = \frac{|-9\hat{i} - 9\hat{k}|}{\sqrt{11}} = \frac{9\sqrt{2}}{\sqrt{11}} = \frac{9\sqrt{22}}{11} \text{ units}$$

MARKING SCHEME

- (i) 1 mark; (ii) 1 mark; (iii) (a) or (b) 2 marks — 1 for the correct cross product and 1 for the correct shortest distance.

Question 37 — 1 + 1 + 2 = 4 marks

Water is essential for life, and rivers are vital water sources across the world. The Yamuna Action Plan (YAP), one of India's largest river restoration projects, was launched to clean the Yamuna, but its effectiveness — the Water Quality Index (WQI) — must be measured through cost-benefit, economic and environmental analysis. Therefore, comprehensive pollutant treatment, prevention of further contamination, and community awareness are crucial for sustainable river management.

Implementing the same concept to the river Yamuna, if an initial base line WQI is 35, a quadratic expression is formed to measure the impact of investment of ₹ x crore on the WQI, i.e. $W(x)$, as

$$W(x) = 35 + 1.6x - 0.02x^2, \text{ where } x \text{ is in rupees in crores}$$

Use the given $W(x)$ to answer the following questions:

- (i) Find the value of x at the stationary point for the function $W(x)$.
- (ii) Find the maximum value of the WQI using the second derivative test.
- (iii)
 - (a) If the investment is ₹35 crore, find by how much the WQI is short of its maximum achievable value.

OR

- (iii)
 - (b) Evaluate the WQI when the investment is ₹60 crore. Determine whether this indicates an enhancement or a diminishing return in our objective when compared with the optimum WQI.

ANSWER AND SOLUTION

$$(i) W(x) = 35 + 1.6x - 0.02x^2 \Rightarrow W'(x) = 1.6 - 0.04x$$

$W'(x) = 0 \Rightarrow 1.6 - 0.04x = 0 \Rightarrow x = 40$, i.e. the stationary point occurs at an investment of ₹40 crore.

$$(ii) W''(x) = -0.04 < 0, \text{ so by the second derivative test } W(x) \text{ is maximum at } x = 40.$$

$$\text{Maximum WQI} = W(40) = 35 + 1.6(40) - 0.02(40)^2 = 35 + 64 - 32 = 67$$

$$(iii) (a) W(35) = 35 + (1.6)(35) - (0.02)(35)^2 = 35 + 56 - 24.5 = 66.5$$

$67 - 66.5 = 0.5$, so the WQI is short by 0.5 units of its maximum level.

$$(iii) (b) W(60) = 35 + (1.6)(60) - (0.02)(60)^2 = 35 + 96 - 72 = 59$$

59 is less than the maximum value 67 of the WQI, even though the investment has been increased from ₹40 crore to ₹60 crore. This shows that the returns are diminishing.

MARKING SCHEME

- (i) 1 mark; (ii) $W''(x) < 0$ with the conclusion — $\frac{1}{2}$, maximum value 67 — $\frac{1}{2}$; (iii) (a) $W(35) = 66.5$ — 1 and the shortfall 0.5 — 1, or (b) $W(60) = 59$ — 1 and the conclusion of diminishing returns — 1.

Question 38 — 2 + 2 = 4 marks

To celebrate their child's birthday, a family decided to distribute fruits among children in three orphanages. The fruit vendor packed three different types of baskets, according to the quantity required at each orphanage; details are given as follows:

Basket A contains 5 kg apples, 10 kg oranges and 5 kg pomegranate.

Basket B contains 10 kg apples, 10 kg oranges and 5 kg pomegranate.

Basket C contains 8 kg apples, 15 kg oranges and 7 kg pomegranate.

Cost of baskets A, B and C are respectively ₹2750, ₹3500 and ₹4100.

Based on the given information, answer the following questions:

- Write the matrix form of the equations involved in this situation and find the adjoint of the coefficient matrix involved.
- Using inverse of the matrix, calculate the price of each fruit per kg.

ANSWER AND SOLUTION

(i) Taking the price of each kg of apples, oranges and pomegranate as ₹ x , ₹ y and ₹ z respectively, the equations are $5x + 10y + 5z = 2750$, $10x + 10y + 5z = 3500$ and $8x + 15y + 7z = 4100$.

The matrix form is $AX = B$, where

$$A = \begin{bmatrix} 5 & 10 & 5 \\ 10 & 10 & 5 \\ 8 & 15 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2750 \\ 3500 \\ 4100 \end{bmatrix}$$

Here $|A| = 25 \neq 0$, so the system is consistent and has a unique solution.

$$\text{adj } A = \begin{bmatrix} -5 & 5 & 0 \\ -30 & -5 & 25 \\ 70 & 5 & -50 \end{bmatrix}$$

(ii) The unique solution is given by $X = A^{-1}B = \frac{1}{25} \begin{bmatrix} -5 & 5 & 0 \\ -30 & -5 & 25 \\ 70 & 5 & -50 \end{bmatrix} \begin{bmatrix} 2750 \\ 3500 \\ 4100 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 150 \\ 100 \\ 200 \end{bmatrix} \Rightarrow x = 150, y = 100, z = 200$$

So the prices of apples, oranges and pomegranate are respectively ₹150 per kg, ₹100 per kg and ₹200 per kg.

MARKING SCHEME

- (i) Writing the equations in matrix form $AX = B$ — 1; finding $\text{adj } A$ — 1. (ii) Writing $X = A^{-1}B$ and multiplying — 1; stating the three prices — 1.

